CONVERGENCE AND MULTIPLICITIES FOR THE LEMPERT FUNCTION

PASCAL J. THOMAS, NGUYEN VAN TRAO

Abstract. Given a domain $\Omega \subset \mathbb{C}^n$, the Lempert function is a functional on the space $Hol(\mathbb{D}, \Omega)$ of analytic disks with values in Ω, depending on a set of poles in Ω. We generalize its definition to the case where poles have multiplicities given by local indicators (in the sense of Rashkovskii) to obtain a function which still dominates the corresponding Green function, behaves relatively well under limits, and is monotonic with respect to the local indicators. In particular, this is an improvement over the previous generalization used by the same authors to find an example of a set of poles in the bidisk so that the (usual) Green and Lempert functions differ.

1. Introduction

We assume throughout that Ω is a bounded domain in $\mathbb{C}^n$. Let $\mathbb{D}$ stand for the unit disk in $\mathbb{C}$. The classical Lempert function with pole at $a \in \Omega$ [9] is defined by

$$\ell_a(z) := \inf \big\{ \log|\zeta| : \exists \varphi \in Hol(\mathbb{D}, \Omega), \varphi(0) = z, \varphi(\zeta) = a \big\}.$$

Given a finite number of points $a_j \in \Omega$, $j = 1, ..., N$, Coman [3] extended this to:

$$(1.1) \quad \ell(z) := \ell_{\{a_1,...,a_N\}}(z) := \inf \Big\{ \sum_{j=1}^{N} \log|\zeta_j| : \\ \exists \varphi \in Hol(\mathbb{D}, \Omega) : \varphi(0) = z, \varphi(\zeta_j) = a_j, j = 1, ..., N \Big\}.$$

The *Green function* for the same poles is

$$g := \sup \{ u \in PSH(\Omega, \mathbb{R}_-) : u(z) \leq \log|z - a_j| + C_j, \\ \text{for } z \text{ in a neighborhood of } a_j, j = 1, ..., N \},$$

where $PSH(\Omega, \mathbb{R}_-)$ stands for the set of all negative plurisubharmonic functions in Ω. The inequality $g(z) \leq \ell(z)$ always holds, and it is

1991 *Mathematics Subject Classification.* [2000] 32U35, 32F45.

Key words and phrases. pluricomplex Green functions, disk functionals, Lempert function, local indicators, Monge-Ampère operator.

known that it can be strict [2], [16], [11]. If ℓ ever turns out to be plurisubharmonic itself, then ℓ must be equal to g [3].

There are natural extensions of the definition of the Green function. In one dimension, considering a finite number of poles in the same location a, say m poles, has a natural interpretation in terms of multiplicities: the point mass in the Riesz measure of the Green function is multiplied by m. Locally, the Green function behaves like $\log|f|$, where f is a holomorphic function vanishing at a with multiplicity m.

Lelong and Rashkovskii [8], [12] defined a generalized Green function. The function $\log|z|$ was replaced by "local indicators", i.e. circled plurisubharmonic functions Ψ whose Monge Ampère measure $(dd^c\Psi)^n$ is concentrated at the origin, such that whenever $\log|w_j| = c\log|z_j|$ for all $j \in \{1,\dots,n\}$, then $\Psi(w) = c\Psi(z)$. This has the advantage of allowing the consideration of non-isotropic singularities such as $\max(2\log|z_1|, \log|z_2|)$, but the "circled" condition privileges certain coordinate axes, so that the class isn't invariant under linear changes of variables. We will have to remove this restriction to obtain a class large enough to describe some natural limits.

In several complex variables, we would like to know which notion of multiplicity can arise when we take limits of ordinary Green (or Lempert) functions with several poles tending to the same point. This idea was put to use in [16] to exhibit an example where a Lempert function with four poles is different from the corresponding Green function. The definition of a generalized Lempert function chosen in [16] had some drawbacks — essentially, it was not monotonic with respect to its system of poles (in an appropriate sense) [16, Proposition 4.3] and did not pass to the limit in some very simple situations [15, Theorem 6.3]. We recall that monotonicity holds when no multiplicities are present, see [18] and [16, Proposition 3.1] for the convex case, and the more recent [10] for arbitrary domains and weighted Lempert functions, or more generally when a subset of the original set of poles is considered with the same generalized local indicators.

In section 2, we successively define a class of indicators, a subclass which is useful to produce "monomial" examples, a notion of multiplicity for values attained by an analytic disk, and a generalization of Coman's Lempert function to systems of poles with generalized local indicators, different from [16]. In section 3, we state our two main results: monotonicity, and convergence under certain restrictive (but, we hope, natural) conditions. Further sections are devoted to the proofs of those results.

Finally, in Section 7 we summarize the differences between our new definition and that given in [16].

The first named author would like to thank Nikolai Nikolov for stimulating discussions on this topic, and his colleague Anne Bauval from Toulouse for showing him a nice purely combinatorial proof of Lemma 5.6. Special thanks are due to the referee for his very thorough reading of our paper.

2. Definitions

Definition 2.1. [8] *Let $\Psi \in PSH(\mathbb{D}^n)$. We call Ψ a* local indicator *and write $\Psi \in \mathcal{I}_0$ if*

1. *Ψ is bounded from above on $\mathbb{D}^n$;*
2. *Ψ is circled, i.e. $\Psi(z_1, \dots, z_n)$ depends only on $(|z_1|, \dots, |z_n|)$;*
3. *for any $c > 0$, $\Psi(|z_1|^c, \dots, |z_n|^c) = c\Psi(|z_1|, \dots, |z_n|)$.*

As a consequence, $(dd^c\Psi)^n = \tau_\Psi \delta_0$ for some $\tau_\Psi \geq 0$.

Notice that if $\Psi_1 \in PSH(\mathbb{D}^n)$, $\Psi_2 \in PSH(\mathbb{D}^m)$, and they are both local indicators, then

$$\Psi(z, z') := \max(\Psi_1(z), \Psi_2(z'))$$

defines a local indicator on $\mathbb{D}^{n+m}$.

We need to remove the restriction to a single coordinate system in Definition 2.1.

Definition 2.2. *We call Ψ a* generalized local indicator*, and we write $\Psi \in \mathcal{I}$ if there exists U a neighborhood of 0, $\Psi_0 \in \mathcal{I}_0$ and a one-to-one linear map L of $\mathbb{C}^n$ to itself such that $L(U) \subset \mathbb{D}^n$ and $\Psi = \Psi_0 \circ L$.*

We will concentrate on a class of simple examples. Given two vectors $z, w \in \mathbb{C}^n$, their standard hermitian product is denoted by $z \cdot \bar{w} := \sum_j z_j \bar{w}_j$. We also write $\|z\| := |z \cdot \bar{z}|^{1/2}$.

Definition 2.3. *We say that Ψ is an* elementary local indicator *if there exists a basis $\{v_1, \dots, v_n\}$ of vectors of $\mathbb{C}^n$ and scalars $m_j \in \mathbb{R}_+$, $1 \leq j \leq n$, such that for $z \in \mathbb{D}^n$,*

$$\Psi(z) = \max_{1 \leq j \leq n} m_j \log |z \cdot \bar{v}_j|. \tag{2.1}$$

One easily checks that any elementary local indicator is a generalized local indicator. The most interesting case is the one for which the basis is orthornormal. In fact, it is essentially the only case.

Lemma 2.4. *Given an elementary local indicator Ψ as in Definition 2.3 there exists an orthonormal basis $\{\tilde{v}_1, \dots, \tilde{v}_n\}$ of $\mathbb{C}^n$ such that the associated elementary local indicator $\tilde{\Psi}(z) := \max_{1 \leq j \leq n} m_j \log |z \cdot \overline{\tilde{v}_j}|$ verifies $\tilde{\Psi} - \Psi \in L^\infty(\mathbb{D}^n)$.*

As a consequence, we could have restricted the map L in Definition 2.2 to be unitary, and it would not have changed things in any essential way.

The proof of Lemma 2.4 is given in Section 4 below.

Lemma 2.5. [8, example in Section 3], [12] *If Ψ is an elementary local indicator, then $\tau_\Psi = m_1 \cdots m_n$.*

We take the same definition of the generalized Green function as in [8].

Definition 2.6. *Let Ω be a bounded domain in $\mathbb{C}^n$. Given*

$$S := \{(a_j, \Psi_j), 1 \leq j \leq N\}, \text{ where } a_j \in \Omega, a_j \neq a_k \text{ for } j \neq k, \Psi_j \in \mathcal{I},$$

its Green function *is*

$$G_S := \sup \{u \in PSH(\Omega, \mathbb{R}_-) : u(z) \leq \Psi_j(z) + C_j, \quad \text{for } z \text{ in a neighborhood of } a_j, j = 1, ..., N\}.$$

To generalize the Lempert function, the first step is to quantify the way in which an analytic disk, i.e. an element of $Hol(\mathbb{D}, \Omega)$, meets a pole provided with a generalized local indicator.

Definition 2.7. *Let $\alpha \in \mathbb{D}$, $a \in \Omega$, $\Psi \in \mathcal{I}$. Then the* multiplicity *of $\varphi \in Hol(\mathbb{D}, \Omega)$ at α, with respect to a, is given by*

$$\text{If } \varphi(\alpha) = a, \quad \text{then} \quad m_{\varphi,a,\Psi}(\alpha) := \min\left(\tau_\Psi, \liminf_{\zeta \to 0} \frac{\Psi(\varphi(\alpha+\zeta) - a)}{\log|\zeta|}\right);$$

$$\text{if } \varphi(\alpha) \neq a, \quad \text{then} \quad m_{\varphi,a,\Psi}(\alpha) := 0.$$

Notice that if $\Psi_1 - \Psi_2$ is locally bounded near the origin, then $m_{\varphi,a,\Psi_1}(\alpha) = m_{\varphi,a,\Psi_2}(\alpha)$.

The quantity $\liminf_{\zeta \to 0} \frac{\Psi(\varphi(\alpha+\zeta)-a)}{\log|\zeta|}$ is exactly the Lelong number at 0 of the subharmonic function $\Psi \circ \varphi$, compare with [13, pp. 334–335]. Truncating at the level of the local Monge-Ampère mass τ_Ψ will turn out to be convenient in Definition 2.8, and the proofs that use it.

It is useful to see what this means in the case of elementary local indicators.

Elementary examples.

Suppose that $\alpha = 0$, $a = 0$, and that $\Psi(z) = \max_{1 \leq j \leq n} m_j \log|z_j|$. We write

$$\varphi(\zeta) = (\varphi_1(\zeta), \ldots, \varphi_n(\zeta)),$$

and define the valuations

$$\nu_j := \nu_j(0, \varphi) := \min\{k : (\frac{d}{d\zeta})^k \varphi_j(0) \neq 0\}.$$

Then we have

$$m_{\varphi,0,\Psi}(0) = \min\left(\min_{1\le j\le n} m_j\nu_j, \prod_{j=1}^{n} m_j\right). \tag{2.2}$$

Example 1.
If $m_j = 1$ for all j, $m_{\varphi,0,\Psi}(0) = 1$ if $\varphi(0) = 0$, $m_{\varphi,0,\Psi}(0) = 0$ otherwise. This is the basic case where one just records whether a point has been hit by the analytic disk or not.

Example 2.
In more general cases, the use of an elementary local indicator will impose higher-order differential conditions on the map φ. For instance, if $m_1 = 2$ and $m_j = 1$, $2 \le j \le n$, then

$$\begin{aligned} m_{\varphi,0,\Psi}(0) &= 0 \text{ if } \varphi(0) \ne 0; \\ m_{\varphi,0,\Psi}(0) &= 1 \text{ if } \varphi(0) = 0 \text{ and } \varphi_j'(0) \ne 0 \text{ for some } j \in \{2,\dots,n\}; \\ m_{\varphi,0,\Psi}(0) &= 2 \text{ if } \varphi(0) = 0 \text{ and } \varphi_j'(0) = 0 \text{ for any } j \in \{2,\dots,n\}. \end{aligned}$$

Definition 2.8. *Given a system S as in Definition 2.6, we write $\tau_j := \tau_{\Psi_j}$.*
Let $\varphi \in Hol(\mathbb{D},\Omega)$ and $A_j \subset \mathbb{D}$, $1 \le j \le N$. We say that $(\varphi,(A_j)_{1\le j\le N})$ is admissible (for S, z) *if*

$$\varphi(0) = z; \quad A_j \subset \varphi^{-1}(a_j) \text{ and } \sum_{\alpha\in A_j} m_{\varphi,a_j,\Psi_j}(\alpha) \le \tau_j, 1 \le j \le N.$$

In this case, we write (with the convention that $0\cdot\infty = 0$)

$$\mathcal{S}(\varphi,(A_j)_{1\le j\le N}) := \sum_{j=1}^{N}\sum_{\alpha\in A_j} m_{\varphi,a_j,\Psi_j}(\alpha)\log|\alpha|.$$

Then the generalized Lempert function is defined by

$$\begin{aligned} &\mathcal{L}_S^{\Omega}(z) := \mathcal{L}_S(z) \\ &\quad := \inf\left\{\mathcal{S}(\varphi,(A_j)_{1\le j\le N}) : (\varphi,(A_j)_{1\le j\le N}) \textit{ is admissible for } S, z\right\}. \end{aligned}$$

Notice that we allow any of the A_j to be the empty set (in which case the j-th term drops from the sum).

Consider the *single poles case* where

$$\text{for each } j, \quad \Psi_j(z) = \max_{1\le l\le n}\log|z_l|, \text{ or } \Psi_j(z) = \log\|z\| \tag{2.3}$$

– it is the same, since both functions differ by a bounded term near 0; in fact, one could use any norm that is homogenous under complex scalar multiplication.

In this case, $\tau_j = 1$ for every j. With a slight abuse of notation, we write $S = \{a_1, \ldots, a_N\}$. Then $\mathcal{L}_S(z) = \min_{S' \subset S} \ell_{S'}(z)$, where ℓ_S is defined in (1.1). And in fact $\min_{S' \subset S} \ell_{S'}(z) = \ell_S(z)$ [10] (see also [17], [18] for the case when the domain Ω is convex).

The Lempert function is different from the functionals considered by Poletsky and others in that it is restricted to one pre-image per pole a_j (thus the Lempert function can fail to be equal to the corresponding Green function). In our definition, the number of pre-images per pole is bounded above by the Monge-Ampère mass at that pole of its generalized local indicator. In [16], each pole only could have one pre-image, but (essentially) φ had to hit the pole with maximum multiplicity at that pre-image.

Although Definition 2.8 may seem contrived, it is required to obtain the reasonable convergence theorem 3.3. See the discussion in Section 7.

We remark right away that the usual relationship holds between this generalized Lempert function and the corresponding Green function.

Lemma 2.9. *For Ω a bounded domain, for any system S as in Definition 2.6, for any $z \in \Omega$, $G_S(z) \le \mathcal{L}_S(z)$.*

Proof. If $\varphi \in Hol(\mathbb{D}, \Omega)$, and $u \in PSH_-(\Omega)$ is a member of the defining family for the Green function of S, then $u \circ \varphi$ is subharmonic and negative on $\mathbb{D}$. Furthermore, if $(\varphi, (A_j)_{1 \le j \le N})$ is admissible (for S, z) and $\alpha \in A_j$, then given any $\varepsilon > 0$, for $|\zeta|$ small enough,

$$u \circ \varphi(\alpha + \zeta) \le C_j + \Psi_j(\varphi(\alpha + \zeta) - a_j) \le C_j + (m_{\varphi, a_j, \Psi_j}(\alpha) - \varepsilon) \log |\zeta|.$$

So $u \circ \varphi$ is a member of the defining family for the Green function on $\mathbb{D}$ with poles α and weights $m_{\varphi, a_j, \Psi_j}(\alpha) - \varepsilon$ at α. This implies that

$$u \circ \varphi(\zeta) \le \sum_{j=1}^{N} \sum_{\alpha \in A_j} (m_{\varphi, a_j, \Psi_j}(\alpha) - \varepsilon) \log \left| \frac{\alpha - \zeta}{1 - \zeta \bar{\alpha}} \right|.$$

Letting ε tend to 0 and setting $\zeta = 0$, we get $u(z) \le \mathcal{S}(\varphi, (A_j)_{1 \le j \le N})$.

Passing to the supremum over u, then to the infimum over $(\varphi, (A_j)_{1 \le j \le N})$, we get the Lemma. $\square$

3. Main Results

We start with a remark.

Lemma 3.1. *If S is as in Definition 2.6, $1 \le N' \le N$, and*

$$S' := \{(a_j, \Psi_j), 1 \le j \le N'\},$$

then for any $z \in \Omega$, $\mathcal{L}_{S'}(z) \ge \mathcal{L}_S(z)$.

Proof. If we take $A_j = \emptyset$ for $N'+1 \le j \le N$, any member of the defining family for $\mathcal{L}_{S'}(z)$ becomes a member of the defining family for $\mathcal{L}_S(z)$, and the sum remains the same. $\square$

The above lemma goes in the direction of monotonicity of the Lempert function with respect to its system of poles. For the Green function, it is immediate that the more poles there are, the more negative the function must be. More generally the more negative the generalized local indicators are (removing a pole corresponds to replacing a local indicator by 0), the more negative the function must be. This is not immediately apparent in Definition 2.8, but it does hold for elementary local indicators.

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{C}^n$,*

$S := \{(a_j, \Psi_j), 1 \le j \le N\}, S' := \{(a_j, \Psi'_j), 1 \le j \le N\},$ *where $a_j \in \Omega$, and Ψ_j, Ψ'_j, are elementary local indicators such that $\Psi_j \le \Psi'_j + C_j$ in a neighborhood of* 0, *$C_j \in \mathbb{R}$, $1 \le j \le N$. Then $\mathcal{L}_{S'}(z) \ge \mathcal{L}_S(z)$, for all $z \in \Omega$.*

The proof is given in Section 5.

Now we turn to a result about the convergence of some families of (ordinary) Lempert functions with single poles, whose limits can be described naturally as generalized Lempert functions. Note that the proof of this next theorem doesn't require the relatively difficult Theorem 3.2, only the easy Lemma 3.1.

For $z \in \mathbb{C}^n \setminus \{0\}$, we denote by $[z]$ the equivalence class of z in the complex projective space $\mathbb{P}^{n-1}$.

Theorem 3.3. *Let Ω be a bounded and convex domain in $\mathbb{C}^n$. Let $0 \le M \le N$ be integers. For ε belonging to a neighborhood of* 0 *in $\mathbb{C}$, using the simplified notation of the single pole case* (2.3)*, let*

$$S(\varepsilon) := \left\{a_j(\varepsilon), 1 \le j \le M; a'_j(\varepsilon), a''_j(\varepsilon), M+1 \le j \le N\right\} \subset \Omega.$$

Suppose that all the points of $S(\varepsilon)$ are distinct for any fixed ε, that

$$\lim_{\varepsilon \to 0} a_j(\varepsilon) = a_j \in \Omega, 1 \le j \le M;$$

$$\lim_{\varepsilon \to 0} a'_j(\varepsilon) = \lim_{\varepsilon \to 0} a''_j(\varepsilon) = a_j \in \Omega, M+1 \le j \le N;$$

and that

$$\lim_{\varepsilon \to 0} [a''_j(\varepsilon) - a'_j(\varepsilon)] = [v_j], \tag{3.1}$$

where the limit is with respect to the distance in $\mathbb{P}^{n-1}$ and the representative v_j is chosen of unit norm. Let $\Psi_j(z) := \log \|z\|$, $1 \le j \le M$.

Denote by π_j *the orthogonal projection onto* $\{v_j\}^\perp$, $M+1 \le j \le N$, *and by* Ψ_j *the generalized local indicator*

$$\Psi_j(z) := \max(\log \|\pi_j(z)\|, 2\log|z\cdot \bar{v}_j|), \quad M+1 \le j \le N.$$

Set $S := \{(a_j, \Psi_j), 1 \le j \le N\}$. *Then*

$$\lim_{\varepsilon\to 0} \ell_{S(\varepsilon)}(z) = \lim_{\varepsilon\to 0} \mathcal{L}_{S(\varepsilon)}(z) = \mathcal{L}_S(z) \text{ for all } z \in \Omega.$$

Remarks : (a) as in the comments after (2.3), one could replace Ψ_j by an elementary local indicator; (b) the convexity requirement is imposed by Lemma 6.1, and we conjecture that it is not essential.

Note that in the case where $a_j'(\varepsilon) = a_j$ does not depend on ε, the hypothesis (3.1) means that the point $a_j''(\varepsilon)$ converges to a limit in the blow-up of $\mathbb{C}^n$ around the point a_j.

It seems to us that this is the only reasonable convergence result that can be obtained for a family of ordinary Lempert functions. If (3.1) is not satisfied, one can find two distinct limit points for our family of Lempert functions. Thus hypothesis (3.1) is required.

We are restricting ourselves to the case where no more than two points converge to the same point: examples where three points converge to the origin in the bidisk are explicitly studied in [14], and show that the situation leads to results that probably can't be described in terms of our generalized local indicators.

The proof is given in Section 6.

4. Proof of Lemma 2.4

Multiplying one of the vectors v_j by a scalar only modifies the function Ψ by a bounded additive term, so it will be enough to exhibit an orthogonal basis of vectors complying with the conclusion of the Lemma.

Renumber the vectors v_j so that we have $0 \le m_1 \le \cdots \le m_n$. Using the Gram-Schmidt orthogonalization process, we produce an orthogonal system of vectors $\tilde{v}_k$ such that $\mathrm{Span}(\tilde{v}_1, \dots, \tilde{v}_k) = \mathrm{Span}(v_1, \dots, v_k)$ for any k, $1 \le k \le n$.

We proceed by induction on the dimension n. When $n = 1$ the property is immediate. Assume that the result holds up to dimension $n-1$. Write

$$\Psi_1(z) := \max_{1\le j\le n-1} m_j \log|z\cdot \bar{v}_j|, \quad \tilde{\Psi}_1(z) := \max_{1\le j\le n-1} m_j \log|z\cdot \overline{\tilde{v}_j}|.$$

Denote $z_n := z\cdot \overline{\tilde{v}_n}$.

It is enough to obtain the estimates on a neighborhood U of 0. We choose it so that for $z \in U$, $|z_n| \le 0$, $\Psi_1(z), \tilde{\Psi}_1(z) \le 0$. Since $v_n =$

$\tilde v_n - w$, where $w \in \operatorname{Span}(v_1, \dots, v_{n-1})$, we have

$$\Psi(z) = \max(\Psi_1(z'), m_n \log|z_n - z' \cdot \bar w|), \qquad (4.1)$$
$$\tilde\Psi(z) = \max(\tilde\Psi_1(z'), m_n \log|z_n|),$$

where z' stands for the orthogonal projection of z on $\operatorname{Span}(v_1, \dots, v_{n-1})$ $= \operatorname{Span}(\tilde v_1, \dots, \tilde v_{n-1})$. By the induction hypothesis, $\Psi_1 = \tilde\Psi_1 + O(1)$, so it is enough to prove that

$$\Psi'(z) := \max(\Psi_1(z'), m_n \log|z_n|)$$

differs from $\Psi(z)$ by a bounded additive term.

There is a constant $C_0 > 0$ such that $\Psi_1(z') \geq m_{n-1} \log\|z'\| - \log C_0$, for $z' \in U$. Choose a constant $A > 1$ large enough so that $\|w\|(C_0/A)^{1/m_{n-1}} < 1/2$.

Then, since $\Psi_1(z) \leq 0$ and $m_{n-1} \leq m_n$,

$$|z' \cdot \bar w| \leq \|w\| C_0^{1/m_{n-1}} \exp(\frac{\Psi_1(z')}{m_{n-1}}) \qquad (4.2)$$
$$\leq \|w\| C_0^{1/m_{n-1}} \exp(\frac{\Psi_1(z')}{m_n}).$$

Case 1. $\Psi_1(z') \geq m_n \log|z_n| - \log A$.

By the inequality above, $\Psi'(z) \leq \Psi_1(z') + \log A \leq \Psi(z) + \log A$.

On the other hand, using (4.2), we get

$$|z_n - z' \cdot \bar w|^{m_n} \leq \left(A^{1/m_{n-1}} + \|w\| C_0^{1/m_{n-1}}\right)^{m_n} \exp(\Psi_1(z')),$$

so $\Psi(z) \leq \Psi_1(z') + O(1) \leq \Psi'(z) + O(1)$.

Case 2. $\Psi_1(z') \leq m_n \log|z_n| - \log A$.

Then (4.2) and the choice of A imply

$$|z' \cdot \bar w| \leq \|w\| C_0^{1/m_{n-1}} \exp\left(\log|z_n| - \frac{\log A}{m_{n-1}}\right) \leq \frac{1}{2}|z_n|,$$

thus (4.1) implies that

$$\Psi'(z) + \log\frac{1}{2} \leq \Psi(z) \leq \Psi'(z) + \log\frac{3}{2}.$$

5. Proof of Theorem 3.2

Without loss of generality, we may assume that $\tau_j' > 0$ for all j.

We have $\Psi_j \leq \Psi_j' + C_j$ in a neighborhood of 0 and

$$\operatorname{supp}(dd^c\Psi_j)^n \subset \{0\}, \quad \operatorname{supp}(dd^c\Psi_j')^n \subset \{0\}.$$

Thus it follows from Bedford and Taylor's comparison theorem [1], [7, p. 126, Theorem 3.7.1] that $\tau_j \geq \tau'_j > 0$. For any α, a_j,

$$m_{\varphi,a_j,\Psi_j}(\alpha) \geq m_{\varphi,a_j,\Psi'_j}(\alpha). \tag{5.1}$$

Therefore

$$\sum_{j=1}^{N} \sum_{\alpha\in A_j} m_{\varphi,a_j,\Psi_j}(\alpha) \log|\alpha| \leq \sum_{j=1}^{N} \sum_{\alpha\in A_j} m_{\varphi,a_j,\Psi'_j}(\alpha) \log|\alpha|. \tag{5.2}$$

To finish the proof, it suffices to show that the family over which we take the infimum is smaller for $\mathcal{L}_{S'}(z)$ than the one for $\mathcal{L}_S(z)$. This can be checked for each j separately, hence we drop the index j.

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{C}^n$. If Ψ, Ψ' are elementary local indicators such that $\Psi \leq \Psi' + C$ and $\tau' := \tau_{\Psi'} > 0$, if $A \subset \mathbb{D}$, $a \in \Omega$ and $\varphi \in Hol(\mathbb{D}, \Omega)$ verify*

$$\sum_{\alpha\in A} m_{\varphi,a,\Psi'}(\alpha) \leq \tau',$$

then

$$\sum_{\alpha\in A} m_{\varphi,a,\Psi}(\alpha) \leq \tau := \tau_{\Psi}.$$

Proof. Since the point a plays no role, we assume $a = 0$ and write $m_{\varphi,0,\Psi}(\alpha) = m_{\varphi,\Psi}(\alpha)$. By (5.1), we may assume that $m_{\varphi,\Psi}(\alpha) > 0$ and the sums in (5.2) will not change.

Using Lemma 2.4, we reduce ourselves to the case where the elementary local indicators are given by orthonormal systems of vectors. We use the same "valuations" as in the Elementary Example:

$$\nu_j(\alpha) := \nu_j(\alpha, \varphi) := \min\{k : (\frac{d}{d\zeta})^k(\varphi(\zeta) \cdot \bar{v}_j)(\alpha) \neq 0\},$$

and $\nu'_j(\alpha)$ is defined analogously using the vectors v'_j.

Case 1. There exists α_0 such that $m_{\varphi,\Psi'}(\alpha_0) = \tau'$.

Then the hypothesis of Lemma 5.1 implies that for all $\alpha \in A \setminus \{\alpha_0\}$, $m_{\varphi,\Psi'}(\alpha) = 0$, so that $\min_{1\leq k\leq n} m'_k\nu'_k(\alpha) = 0$. Since $\tau' > 0$, we have $m'_k > 0$ for all k, so there must exist k such that $\nu'_k(\alpha) = 0$. Then $\varphi(\alpha) \neq 0$, which implies that $m_{\varphi,\Psi}(\alpha) = 0$, and

$$\sum_{\alpha\in A} m_{\varphi,\Psi}(\alpha) = m_{\varphi,\Psi}(\alpha_0) \leq \tau,$$

by definition of the multiplicity.

Case 2. For all $\alpha \in A$, $m_{\varphi,\Psi'}(\alpha) < \tau'$.

Therefore $m_{\varphi,\Psi'}(\alpha) = \min_{1\leq k\leq n} m'_k\nu'_k(\alpha)$, and since we always have $m_{\varphi,\Psi}(\alpha) \leq \min_{1\leq k\leq n} m_k\nu_k(\alpha)$, it becomes enough to work with those

quantities in (5.2). By dividing by τ and τ' respectively, it will be enough to prove the following Lemma. $\square$

Lemma 5.2. *Under the hypotheses of Lemma 5.1 and Case 2 above, for each $\alpha \in A$,*

$$\frac{\min_{1\le k\le n} m'_k \nu'_k(\alpha)}{\prod_{k=1}^n m'_k} \ge \frac{\min_{1\le k\le n} m_k \nu_k(\alpha)}{\prod_{k=1}^n m_k}.$$

Proof. Since we are now dealing with a single α, we also drop it from the notation.

We introduce a binary relation on the index set $\{1,\dots,n\}$.

Definition 5.3. *Given $k,l \in \{1,\dots,n\}$, we say that $k\mathcal{R}l$ if and only if $v_k \cdot \bar{v}'_l \neq 0$.*

Lemma 5.4. *If $\Psi' + C \ge \Psi$ and $k\mathcal{R}l$, then $m_k \ge m'_l$.*

Proof. For any nonzero $\lambda \in \mathbb{C}$,

$$\Psi'(\lambda v'_l) = m'_l \log|\lambda| + m'_l \log \|v'_l\|^2,$$

while, for $|\lambda|$ small enough,

$$\Psi(\lambda v'_l) = \max_{1\le j\le n} (m_j(\log|\lambda| + \log|v_j \cdot \bar{v}'_l|)) = (\min_{k:k\mathcal{R}l} m_k)\log|\lambda| + O(1),$$

therefore by letting λ tend to 0 we see that $\min_{k:k\mathcal{R}l} m_k \ge m'_l$. $\square$

Lemma 5.5. *If $\Psi' + C \ge \Psi$, then*

(1) $\nu'_l \ge \min\{\nu_k : k\mathcal{R}l\}$,
(2) $\nu_k \ge \min\{\nu'_l : k\mathcal{R}l\}$.

Proof. We will use and prove part (1) only. The other one has a similar proof.

Since v'_l is orthogonal to v_k unless $k\mathcal{R}l$, we must have complex scalars c_k such that $v'_l = \sum_{k:k\mathcal{R}l} c_k v_k$, thus for φ as in Lemma 5.1,

$$\varphi(\zeta)\cdot \bar{v}'_l = \sum_{k:k\mathcal{R}l} \bar{c}_k \varphi(\zeta)\cdot \bar{v}_k.$$

Now take $m < \nu_k = \nu_k(\alpha,\varphi)$, for all k such that $k\mathcal{R}l$. Then

$$(\frac{d}{d\zeta})^m(\varphi\cdot\bar{v}'_l)(\alpha) = \sum_{k:k\mathcal{R}l} \bar{c}_k (\frac{d}{d\zeta})^m(\varphi(\zeta)\cdot\bar{v}_k)(\alpha) = 0,$$

so we must have $\nu'_l > m$, which proves the result. $\square$

Now renumber the vectors v'_l so that $\min_k(m'_k\nu'_k) = m'_1\nu'_1$. Pick an index k_0 such that $k_0\mathcal{R}1$ and $\nu_{k_0} = \min\{\nu_k : k\mathcal{R}1\}$. By renumbering the vectors v_k, we may assume $k_0 = 1$. By Lemma 5.5, we may assume $\nu'_1 \ge \nu_1$.

The conclusion of Lemma 5.2 thus reduces to:

$$\frac{\nu_1'}{\prod_{k=2}^n m_k'} \geq \frac{\nu_1}{\prod_{k=2}^n m_k}. \tag{5.3}$$

This is a consequence of the next result.

Lemma 5.6. *There exists a bijection σ from $\{2,\dots,n\}$ onto itself such that for any $l \in \{2,\dots,n\}$, $\sigma(l)\mathcal{R}l$.*

This Lemma will be proved below. It implies

$$\prod_{k=2}^n m_k = \prod_{l=2}^n m_{\sigma(l)} \geq \prod_{l=2}^n m_l',$$

by Lemma 5.4, so (5.3) holds and this concludes the proof of Lemma 5.2. □

Proof of Lemma 5.6

Denote $A := (a_{kl})_{2\leq k,l\leq n} := (v_k \cdot \overline{v_l'})_{2\leq k,l\leq n}$. First we prove that this matrix is non singular. Let π be the orthogonal projection on $\{v_1'\}^\perp$. If rank$\{\pi(v_k), 2 \leq k \leq n\} < n-1$, there exists $w \in \{v_1'\}^\perp$, $w \neq 0$, such that $w \perp \pi(v_k)$, $2 \leq k \leq n$. This implies $w \perp v_k$, $2 \leq k \leq n$. Since we have orthogonal bases, $v_1 = \lambda w$, for some $\lambda \in \mathbb{C}$. So $v_1 \cdot \overline{v_1'} = 0$, which contradicts the fact that $1\mathcal{R}1$.

We construct the bijection σ by induction on n. For $n = 2$ it's obvious. Suppose that the property holds for $n-1$. Then

$$0 \neq \det A = \sum_{k=2}^n (-1)^k a_{k2} \det A_k,$$

where A_k stands for the minor matrix with the first column and the k-th row removed. There must be some k for which $a_{k2}\det A_k \neq 0$. Let $\sigma(2) = k$; the induction hypothesis gives us a bijection σ' from $\{3,\dots,n\}$ to $\{2,\dots,n\}\setminus\{k\}$ such that $a_{\sigma'(l)l} \neq 0$, and this finishes the proof. □

6. Proof of Theorem 3.3

First observe that we can relax the conditions used in Definition 2.8.

Lemma 6.1. *Let Ω be a convex bounded domain in $\mathbb{C}^n$ containing the origin, and let $z \in \Omega$.*

(i) Let $a_j \in \Omega$, $\Psi_j \in \mathcal{I}$ and, as in Definition 2.6

$$S := \{(a_j, \Psi_j), 1 \leq j \leq N\}.$$

Suppose that for any $\delta > 0$, there exists a map φ^δ holomorphic from $\mathbb{D}$ to $(1+\delta)\Omega$ and sets $(A_j(\delta))_{1\le j\le N}$ such that $(\varphi^\delta, (A_j(\delta))_{1\le j\le N})$ is admissible for S, z with respect to $(1+\delta)\Omega$ and

$$\mathcal{S}(\varphi^\delta, (A_j(\delta))_{1\le j\le N}) \le \ell + h(\delta),$$

where $h(\delta) \ge 0$, $\lim_{\delta\to 0} h(\delta) = 0$. Then $\mathcal{L}_S^\Omega(z) \le \ell$.

(ii) For ε in a neighborhood V of 0 in $\mathbb{C}$, let $a_j(\varepsilon) \in \Omega$, $1 \le j \le N$.

$$S(\varepsilon) := \{(a_j(\varepsilon), \Psi_j), 1 \le j \le N\}.$$

Let $g : V \longrightarrow \mathbb{R}_+^*$ be such that $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$. Then

$$\limsup_{\varepsilon\to 0} \mathcal{L}_{S(\varepsilon)}^\Omega(z) \le \limsup_{\varepsilon\to 0} \mathcal{L}_{S(\varepsilon)}^{(1+g(\varepsilon))\Omega}(z).$$

Proof. Without loss of generality, we may assume $z = 0$.

Let

$$\Omega_r := \{\varphi(\zeta) : \varphi \in Hol(\mathbb{D}, \Omega), \varphi(0) = 0, |\zeta| < r\}.$$

A bounded convex domain is Kobayashi complete hyperbolic [4, Proposition 6.9 (b), p. 88], so Ω_r is relatively compact in Ω. Let ρ_Ω stand for the Minkowski function of Ω:

$$\rho_\Omega(z) := \inf\{r > 0 : \frac{z}{r} \in \Omega\}.$$

We set $\gamma_\Omega(r) := \sup_{\Omega_r} \rho_\Omega$. The function γ is increasing and continuous from $(0,1)$ to itself.

For any $\mu \in (0,1)$, $\phi \in Hol(\mathbb{D}, \mathbb{C}^n)$, denote $\phi_\mu(\zeta) := \phi(\mu\zeta)$. Note that for any points and generalized local indicators, $m_{\phi_\mu,a,\Psi}(\alpha/\mu) = m_{\phi,a,\Psi}(\alpha)$.

Take φ^δ as in Part (i) of the Lemma, in particular $\varphi^\delta(0) = 0$, so by construction of γ,

$$\frac{1}{(1+\delta)}\varphi_\mu^\delta(\mathbb{D}) \subset \gamma(\mu)\Omega.$$

Choose some $\mu(\delta)$ such that $\gamma(\mu(\delta)) = (1+\delta)^{-1}$, and set $\tilde\varphi^\delta := \varphi^\delta_{\mu(\delta)}$, then $\tilde\varphi^\delta \in Hol(\mathbb{D}, \Omega)$. Note that $\lim_{\delta\to 0} \mu(\delta) = 1$, by the relative compactness of each Ω_r.

Let

$$\tilde A_j(\delta) := \left\{ \frac{\alpha}{\mu(\delta)} : \alpha \in A_j(\delta), |\alpha| < \mu(\delta) \right\}.$$

Then

$$(6.1) \quad \left| \mathcal{S}(\tilde{\varphi}^\delta, (\tilde{A}_j(\delta))_{1\le j\le N}) - \mathcal{S}(\varphi^\delta, (A_j(\delta))_{1\le j\le N}) \right|$$

$$= \left| \sum_j \sum_{\alpha\in A_j, |\alpha|<\mu(\delta)} m_{\varphi^\delta, a_j, \Psi_j}(\alpha) |\log\mu(\delta)| - \sum_j \sum_{\alpha\in A_j, |\alpha|\ge\mu(\delta)} m_{\varphi^\delta, a_j, \Psi_j}(\alpha) \log|\alpha| \right|$$

$$\le 2(\sum_j \tau_{\Psi_j}) |\log\mu(\delta)|,$$

and this last quantity tends to 0, which concludes the proof of (i).

To prove (ii), take maps φ^ε and systems of points $(A_j(\varepsilon))$, admissible for $S(\varepsilon)$, such that

$$\lim_{\varepsilon\to 0} \mathcal{S}(\varphi^\varepsilon, (A_j(\varepsilon))_{1\le j\le N}) = \limsup_{\varepsilon\to 0} \mathcal{L}_{S(\varepsilon)}^{(1+g(\varepsilon))\Omega}(0).$$

Use the above proof with $\delta = g(\varepsilon)$ to construct maps $\tilde{\varphi}^\varepsilon$ into Ω and systems of points $(\tilde{A}_j(\varepsilon))$, admissible for $S(\varepsilon)$, such that

$$\left| \mathcal{S}(\tilde{\varphi}^\varepsilon, (\tilde{A}_j(\varepsilon))_{1\le j\le N}) - \mathcal{S}(\varphi^\varepsilon, (A_j(\varepsilon))_{1\le j\le N}) \right| \le 2(\sum_j \tau_{\Psi_j}) |\log\mu(g(\varepsilon))|,$$

and by definition $\mathcal{S}(\tilde{\varphi}^\varepsilon, (\tilde{A}_j(\varepsilon))_{1\le j\le N}) \ge \mathcal{L}_{S(\varepsilon)}^{\Omega}(0)$. □

Consider as in Theorem 3.3 a bounded convex domain Ω, and distinct points $a_j \in \Omega$, $1 \le j \le N$. Let $z \in \Omega \setminus \{a_j, 1 \le j \le N\}$ (otherwise the property is trivially true). Again we may assume $z = 0$. By Lemma 6.1 applied to $S(\delta) = S$ for any δ, to show that

$$(6.2) \qquad \mathcal{L}_S(z) \le \liminf_{\varepsilon\to 0} \mathcal{L}_{S(\varepsilon)}(z) =: \ell,$$

it will be enough to provide: some increasing function g such that $g(0) = 0$ and, for any $\delta > 0$, $\varphi^\delta \in Hol(\mathbb{D}, (1 + g(\delta))\Omega)$ and subsets $(A_j^\delta)_{1\le j\le N}$ of Ω such $(\varphi^\delta, (A_j^\delta)_{1\le j\le N})$ is admissible for S, z and that

$$\mathcal{S}(\varphi^\delta, (A_j^\delta)_{1\le j\le N}) = \ell.$$

The systems $S(\varepsilon)$ all have single poles, so the definition of ℓ means that there exist $\varphi_m \in Hol(\mathbb{D}, \Omega)$, $\varepsilon_m \to 0$, and points $\alpha_{j,m}, \alpha'_{j,m}, \alpha''_{j,m} \in \mathbb{D}$ such that $\varphi_m(\alpha_{j,m}) = a_j(\varepsilon_m)$, $1 \le j \le M$, and $\varphi_m(\alpha'_{j,m}) = a'_j(\varepsilon_m)$, $\varphi_m(\alpha''_{j,m}) = a''_j(\varepsilon_m)$, $M + 1 \le j \le N$; and they satisfy

$$\sum_{j=1}^{M} \log|\alpha_{j,m}| + \sum_{j=M+1}^{N} \left(\log|\alpha'_{j,m}| + \log|\alpha''_{j,m}|\right) = \ell + \delta(m),$$

with $\lim_{m\to\infty} \delta(m) = 0$.

Passing to a subsequence, for which we keep the same notations, we may assume that $\alpha_{j,m} \to \alpha_j \in \overline{\mathbb{D}}$, $\alpha'_{j,m} \to \alpha'_j \in \overline{\mathbb{D}}$, $\alpha''_{j,m} \to \alpha''_j \in \overline{\mathbb{D}}$ as $m \to \infty$, and that $\varphi_m \to \tilde{\varphi} \in Hol(\mathbb{D}, \overline{\Omega})$ uniformly on compact subsets of $\mathbb{D}$. Furthermore, by compactness of the unit circle, there exists $\tilde{v_j} \in [v_j] \cap S^{2n-1}$ such that, taking a further subsequence,

$$\lim_{m\to\infty} \frac{a''_j(\varepsilon_m) - a'_j(\varepsilon_m)}{\|a''_j(\varepsilon_m) - a'_j(\varepsilon_m)\|} = \tilde{v_j}.$$

By renumbering the points and exchanging a'_j and a''_j as needed, we may assume that there are integers $M' \leq M$, $M \leq N_1 \leq N_2 \leq N_3 \leq N$ such that

$$\begin{aligned} \alpha_j \in \mathbb{D} \quad & \text{for} \quad 1 \leq j \leq M' \\ \alpha_j \in \partial\mathbb{D} \quad & \text{for} \quad M'+1 \leq j \leq M \\ \alpha'_j = \alpha''_j \in \mathbb{D} \quad & \text{for} \quad M+1 \leq j \leq N_1 \\ |\alpha'_j| < |\alpha''_j| < 1 \quad & \text{for} \quad N_1+1 \leq j \leq N_2 \\ |\alpha'_j| < 1, |\alpha''_j| = 1 \quad & \text{for} \quad N_2+1 \leq j \leq N_3 \\ |\alpha'_j| = |\alpha''_j| = 1 \quad & \text{for} \quad N_3+1 \leq j \leq N. \end{aligned}$$

Then

$$\ell = \lim_{m\to\infty} \left(\sum_{j=1}^{M} \log|\alpha_{j,m}| + \sum_{j=M+1}^{N} \left(\log|\alpha'_{j,m}| + \log|\alpha''_{j,m}| \right) \right)$$
$$= \sum_{j=1}^{M'} \log|\alpha_j| + \sum_{j=M+1}^{N_1} 2\log|\alpha'_j| + \sum_{j=N_1+1}^{N_2} \left(\log|\alpha'_j| + \log|\alpha''_j| \right) + \sum_{j=N_2+1}^{N_3} \log|\alpha'_j|.$$

Now we choose

$$\begin{aligned} A_j = \{\alpha_j\} \quad & \text{for} \quad 1 \leq j \leq M' \\ A_j = \emptyset \quad & \text{for} \quad M'+1 \leq j \leq M \\ A_j = \{\alpha'_j\} \quad & \text{for} \quad M+1 \leq j \leq N_1 \\ A_j = \{\alpha'_j, \alpha''_j\} \quad & \text{for} \quad N_1+1 \leq j \leq N_2 \\ A_j = \{\alpha'_j\} \quad & \text{for} \quad N_2+1 \leq j \leq N_3 \\ A_j = \emptyset \quad & \text{for} \quad N_3+1 \leq j \leq N. \end{aligned}$$

Notice that $(\tilde{\varphi}, (A_j)_{1\leq j\leq N})$ hits the correct points but doesn't necessarily produce an admissible choice, because for some j, $N_1+1 \leq j \leq N_2$, we could have

$$m_{\tilde{\varphi},a_j,\Psi_j}(\alpha'_j) + m_{\tilde{\varphi},a_j,\Psi_j}(\alpha''_j) > 2 = \tau_j.$$

So, in order to apply Lemma 6.1 with $\delta \to 0$, we set $A_j^\delta = A_j$ for any $\delta > 0$ and

$$\tilde{\varphi}^\delta(\zeta) := \tilde{\varphi}(\zeta) + \delta\zeta \left[\prod_1^{M'} (\zeta - \alpha_j) \prod_{M+1}^{N_1} (\zeta - \alpha'_j)^2 \prod_{N_1+1}^{N_2} (\zeta - \alpha'_j)(\zeta - \alpha''_j) \prod_{N_2+1}^{N_3} (\zeta - \alpha'_j) \right] v,$$

where $v \in \mathbb{C}^n$ is a unit vector chosen such that $\pi_j(v) \neq 0$, $N_1 + 1 \leq j \leq N_3$. For any $\alpha \in \cup_1^N A_j$, $\tilde{\varphi}^\delta(\alpha) = \tilde{\varphi}(\alpha)$. There is a constant $C > 0$ such that $\tilde{\varphi}^\delta(\mathbb{D}) \subset \Omega + C\delta B(0,1)$.

All the following considerations apply when δ is small enough.

For $1 \leq j \leq M'$, $m_{\tilde{\varphi}^\delta, a_j, \Psi_j}(\alpha_j) = 1$, because $\tilde{\varphi}^\delta$ takes on the correct value, and the multiplicity cannot be more than $1 = \tau_j$ in those cases.

For $N_1 + 1 \leq j \leq N_3$, we have

$$\pi_j((\tilde{\varphi}^\delta)'(\alpha'_j)) = \pi_j((\tilde{\varphi})'(\alpha'_j)) + \delta p_j \pi_j(v),$$

where p_j is some complex scalar which doesn't depend on δ, so for $\delta > 0$ and small enough, this projection doesn't vanish and we have $m_{\tilde{\varphi}^\delta, a_j, \Psi_j}(\alpha'_j) = 1$. An analogous reasoning shows that $m_{\tilde{\varphi}^\delta, a_j, \Psi_j}(\alpha''_j) = 1$ for $N_1 + 1 \leq j \leq N_2$.

For $M + 1 \leq j \leq N_1$, we have

$$(\tilde{\varphi}^\delta)'(\alpha'_j) = (\tilde{\varphi})'(\alpha'_j),$$

and by the uniform convergence on compact sets,

$$(\tilde{\varphi})'(\alpha'_j) = \lim_{m\to\infty} \frac{\varphi_m(\alpha'_{j,m}) - \varphi_m(\alpha''_{j,m})}{\alpha'_{j,m} - \alpha''_{j,m}} = \lim_{m\to\infty} \frac{a'_j(\varepsilon_m) - a''_j(\varepsilon_m)}{\alpha'_{j,m} - \alpha''_{j,m}},$$

which must be colinear to v_j by definition. Therefore $m_{\tilde{\varphi}^\delta, a_j, \Psi_j}(\alpha'_j) = 2$ for $M + 1 \leq j \leq N_1$.

Thus $(\tilde{\varphi}^\delta, (A_j)_{1\leq j\leq N})$ is admissible for $S, 0$ and $\mathcal{S}(\tilde{\varphi}^\delta, (A_j)_{1\leq j\leq N}) = \ell$, which proves (6.2).

Now we need to show that

$$\mathcal{L}_S(z) \geq \limsup_{\varepsilon\to 0} \mathcal{L}_{S(\varepsilon)}(z). \tag{6.3}$$

We use Lemma 6.1(ii). For any $\delta > 0$, we need to construct a positive function g such that $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$ and, for ε small enough, $\varphi^\varepsilon \in Hol(\mathbb{D}(1 + g(\varepsilon))\Omega)$ and sets $(A_j(\varepsilon))_{1\leq j\leq N}$ such that $(\varphi^\varepsilon, (A_j(\varepsilon))_{1\leq j\leq N})$ is admissible for $S(\varepsilon), 0$ and

$$\mathcal{S}(\varphi^\varepsilon, (A_j(\varepsilon))_{1\leq j\leq N}) \leq \mathcal{L}_S(z) + \delta.$$

We start with an admissible choice $(\varphi, (A_j)_{1\leq j\leq N})$ for S, such that

$$\mathcal{S}(\varphi, (A_j)_{1\leq j\leq N}) \leq \mathcal{L}_S(z) + \delta/2.$$

To fix notations, suppose that, after renumbering and exchanging the points as needed, there exist integers $M' \leq M$, $N_1, N_2, N_3 \in \{M, \dots N\}$ such that

$$\begin{aligned} A_j = \{\alpha_j\} \quad &\text{for} \quad 1 \leq j \leq M', \\ A_j = \emptyset \quad &\text{for} \quad M'+1 \leq j \leq M, \\ A_j = \{\alpha'_j\}, m_{\varphi,a_j,\Psi_j}(\alpha'_j) = 2 \quad &\text{for} \quad M+1 \leq j \leq N_1, \\ A_j = \{\alpha'_j, \alpha''_j\}, \alpha'_j \neq \alpha''_j \quad &\text{for} \quad N_1+1 \leq j \leq N_2, \\ A_j = \{\alpha'_j\}, m_{\varphi,a_j,\Psi_j}(\alpha'_j) = 1 \quad &\text{for} \quad N_2+1 \leq j \leq N_3, \\ A_j = \emptyset \quad &\text{for} \quad N_3+1 \leq j \leq N. \end{aligned}$$

The definition of Ψ_j (see the computations performed in the Elementary example) implies that, for $M+1 \leq j \leq N_1$, $\varphi'(\alpha'_j) \cdot \bar{w} = 0$, for any $w \in v_j^\perp$. We perturb φ to make sure that, on the other hand, $\varphi'(\alpha'_j) \cdot \bar{v}_j \neq 0$ in the same index range. For $\eta(\varepsilon) \in \mathbb{C}$ to be chosen later, set

$$\begin{aligned} \tilde{\varphi}(\zeta) := \varphi(\zeta) + \\ \eta(\varepsilon) \left[\zeta \prod_1^{M'} (\zeta - \alpha_j) \prod_{j=N_1+1}^{N_2} (\zeta - \alpha'_j)(\zeta - \alpha''_j) \prod_{j=N_2+1}^{N_3} (\zeta - \alpha'_j) \right] \times \\ \times \left\{ \sum_{j=M+1}^{N_1} \left[(\zeta - \alpha'_j) \prod_{M+1 \leq k \leq N_1, k \neq j} (\zeta - \alpha'_k)^2 \right] v_j \right\}. \end{aligned}$$

The map $\tilde{\varphi}$ depends on ε and is admissible again.

We have positive constants C_1, C_2, C_3 such that

- $\tilde{\varphi}'(\alpha'_j) = \lambda_j v_j$, with $C_1^{-1}|\eta(\varepsilon)| \leq |\lambda_j| \leq C_1 |\eta(\varepsilon)|$,
- $\|\tilde{\varphi} - \varphi\|_\infty \leq C_2 |\eta(\varepsilon)|$,
- $\tilde{\varphi}(\mathbb{D}) \subset (1 + C_3|\eta(\varepsilon)|)\Omega$;

in particular $\tilde{\varphi}$ will be bounded by constants independent of ε, along with all its derivatives on any given compact subset of $\mathbb{D}$.

For $M+1 \leq j \leq N$ and ε in a neighborhood of 0, $a''_j(\varepsilon) - a'_j(\varepsilon) = n_j(\varepsilon) v_j(\varepsilon)$, where $\|v_j(\varepsilon)\| = 1$, $\lim_{\varepsilon \to 0} v_j(\varepsilon) = v_j$ and $n_j(\varepsilon) \in \mathbb{C}$.

For $|\varepsilon|$ small enough, we now may define

$$A_j(\varepsilon) := A_j, \text{ for } 1 \leq j \leq M, N_1+1 \leq j \leq N,$$
$$\text{and } A_j(\varepsilon) := \{\alpha'_j, \alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}\}, \text{ for } M+1 \leq j \leq N_1.$$

We shall need to add to $\tilde{\varphi}$ a vector-valued correcting term obtained by Lagrange interpolation. To this end, we write $B(\varepsilon) := \cup_j A_j(\varepsilon)$, and

values to be interpolated, $w(\alpha)$, for $\alpha \in B(\varepsilon)$. Let

$$\begin{aligned}
w(\alpha_j) &:= a_j(\varepsilon) - a_j = a_j(\varepsilon) - \tilde{\varphi}(\alpha_j) && \text{for} \quad 1 \le j \le M', \\
w(\alpha'_j) &:= a'_j(\varepsilon) - a_j = a'_j(\varepsilon) - \tilde{\varphi}(\alpha'_j) && \text{for} \quad M+1 \le j \le N_1, \\
w(\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}) &:= a''_j(\varepsilon) - \tilde{\varphi}(\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}) && \text{for} \quad M+1 \le j \le N_1, \\
w(\alpha'_j) &:= a'_j(\varepsilon) - a_j = a'_j(\varepsilon) - \tilde{\varphi}(\alpha'_j) && \text{for} \quad N_1+1 \le j \le N_2, \\
w(\alpha''_j) &:= a''_j(\varepsilon) - a_j = a''_j(\varepsilon) - \tilde{\varphi}(\alpha''_j) && \text{for} \quad N_1+1 \le j \le N_2, \\
w(\alpha'_j) &:= a'_j(\varepsilon) - a_j = a'_j(\varepsilon) - \tilde{\varphi}(\alpha'_j) && \text{for} \quad N_2+1 \le j \le N_3.
\end{aligned}$$

We denote by P_ε the solution to the interpolation problem

$$(P(\alpha) = w(\alpha) : \alpha \in B(\varepsilon)) .$$

Let $\varphi^\varepsilon := \tilde{\varphi} + P_\varepsilon \in Hol(\mathbb{D}, \Omega^\varepsilon)$. The domain Ω^ε will be specified below. By construction $(\varphi^\varepsilon, (A_j(\varepsilon))_{1\le j\le N})$ is admissible for $S(\varepsilon)$, and for $|\varepsilon|$ small enough,

$$\mathcal{S}(\varphi^\varepsilon, (A_j(\varepsilon))_{1\le j\le N}) \le \mathcal{L}_S(z) + \delta,$$

provided that, for $M+1 \le j \le N_1$,

$$\lim_{\varepsilon\to 0} \frac{n_j(\varepsilon)}{\lambda_j} = 0, \tag{6.4}$$

Now we need to show that the correction is small, more precisely that we can choose $\eta(\varepsilon)$ so that the above condition is satisfied and $\lim_{\varepsilon\to 0} \|P_\varepsilon\|_\infty = 0$. Then we can choose a function g tending to 0 such that

$$\Omega^\varepsilon = (1 + g(\varepsilon))\Omega \supset (1 + C_3|\eta(\varepsilon)|)\Omega + B(0, \|P_\varepsilon\|_\infty).$$

Write Π_α for the unique (scalar) polynomial of degree less or equal to $d := \#B(\varepsilon) - 1$ (d does not depend on ε) such that

$$\Pi_\alpha(\alpha) = 1, \Pi_\alpha(\beta) = 0 \text{ for any } \beta \in B(\varepsilon) \setminus \{\alpha\}.$$

Then

$$P_\varepsilon = \sum_{\alpha\in B(\varepsilon)} \Pi_\alpha w(\alpha).$$

For $\alpha \in \bigcup_{1\le j\le M, N_1+1\le j\le N} A_j$, $\|\Pi_\alpha\|_\infty$ is uniformly bounded, because $\mathrm{dist}(\alpha, B(\varepsilon) \setminus \{\alpha\}) \ge \gamma > 0$ with γ independent of ε. It also follows from the hypotheses of the theorem and the choice of w that

$$\lim_{\varepsilon\to 0} \max\{\|w(\alpha)\|, \alpha \in \bigcup_{1\le j\le M, N_1+1\le j\le N} A_j\} = 0.$$

For $M+1 \le j \le N_1$, we need an elementary lemma about Lagrange interpolation.

Lemma 6.2. *Let $x_0, \dots, x_d \in \mathbb{D}$, $w_0, w_1 \in \mathbb{C}^n$. Suppose that there exists $\gamma > 0$ such that $|x_0 - x_1| \le \gamma$ and dist$([x_0, x_1], \{x_2, \dots, x_d\}) \ge 2\gamma$, where $[x_0, x_1]$ is the real line segment from x_0 to x_1.*

Let P be the unique ($\mathbb{C}^n$-valued) polynomial of degree less or equal to d such that

$$P(x_0) = w_0, P(x_1) = w_1, P(x_j) = 0, 2 \le j \le d.$$

Then there exist constants L_1, L_0 depending only on γ and d such that

$$\sup_{\zeta \in \mathbb{D}} \|P(\zeta)\| \le L_1 \left\| \frac{w_1 - w_0}{x_1 - x_0} \right\| + L_0 \|w_0\|.$$

We will prove this Lemma a little later. It yields, for $M+1 \le j \le N_1$,

$$\begin{aligned} \sup_{\zeta \in \mathbb{D}} & \left\| \Pi_{\alpha'_j}(\zeta) w(\alpha'_j) + \Pi_{\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}}(\zeta) w(\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}) \right\| \\ & \le L_1 \left| \frac{\lambda_j}{n_j(\varepsilon)} \right| \left\| a''_j(\varepsilon) - \tilde{\varphi}(\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}) \right\| + L_0 \|a'_j(\varepsilon) - a_j\|. \end{aligned}$$

We now estimate the first term in the last sum above. By the Taylor formula,

$$a''_j(\varepsilon) - \tilde{\varphi}(\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}) = a''_j(\varepsilon) - a'_j(\varepsilon) - n_j(\varepsilon) v_j + R_2(\varepsilon) = n_j(\varepsilon)(v_j(\varepsilon) - v_j) + R_2(\varepsilon),$$

where $\|R_2(\varepsilon)\| \le C |n_j(\varepsilon)|^2 |\lambda_j|^{-2}$ with C a constant independent of ε by the boundedness of the derivatives of $\tilde{\varphi}$. Finally

$$\begin{aligned} \sup_{\zeta \in \mathbb{D}} & \left\| \Pi_{\alpha'_j}(\zeta) w(\alpha'_j) + \Pi_{\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}}(\zeta) w(\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j}) \right\| \\ & \le C \left(\|v_j(\varepsilon) - v_j\| |\eta(\varepsilon)| + |n_j(\varepsilon)| |\eta(\varepsilon)|^{-1} + \|a'_j(\varepsilon) - a_j\| \right). \end{aligned}$$

To to satisfy (6.4), we need to have $\lim_{\varepsilon \to 0} n_j(\varepsilon)/\eta(\varepsilon) = 0$; to make sure, in addition, that the whole sum above tends to 0 as ε tends to 0, it will be enough to choose $\eta(\varepsilon)$ going to zero, but more slowly that $|n_j(\varepsilon)| = \|a''_j(\varepsilon) - a'_j\|$, for $M+1 \le j \le N_1$. □

Proof of Lemma 6.2 Let

$$Q(X, Y) := \prod_2^d \frac{X - x_k}{Y - x_k}.$$

Then Q and all of its derivatives are bounded for $X \in \overline{\mathbb{D}}$ and $Y \in [x_0, x_1]$.

$$P(X) = \frac{X - x_0}{x_1 - x_0} Q(X, x_1) w_1 + \frac{X - x_1}{x_0 - x_1} Q(X, x_0) w_0$$
$$= \frac{w_1 - w_0}{x_1 - x_0}(X - x_0) Q(X, x_1) + \left(-Q(X, x_1) + (X - x_1) \frac{Q(X, x_1) - Q(X, x_0)}{x_0 - x_1} \right) w_0.$$

Then the conclusion follows from the boundedness of Q and Q' and the mean value theorem. □

7. Comparison with previous results

In [16], we had used a different definition for a Lempert function with multiplicities. We state it with the same notations as in Definition 2.8.

Definition 7.1. *Given a system S as in Definition 2.6, we write $\tau_j := \tau_{\Psi_j}$.*

Let $\varphi \in Hol(\mathbb{D}, \Omega)$ and $\alpha_j \in \mathbb{D}$, $1 \le j \le N$. We say that $(\varphi, (\alpha_j)_{1 \le j \le N})$ is admissible (for S, z) in the old sense *if*

$$\varphi(0) = z, \text{ and there exists } U_j \text{ a neighborhood of } \zeta_j$$
$$\text{s.t. } \Psi_j(\varphi(\zeta) - a_j) \le \tau_j \log|\zeta - \zeta_j| + C_j, \forall \zeta \in U_j, 1 \le j \le N.$$

In this case, we write (with the convention that $0 \cdot \infty = 0$)

$$\mathcal{S}(\varphi, (\alpha_j)_{1 \le j \le N}) := \sum_{j=1}^{N} \tau_j \log|\alpha_j|.$$

Then the old generalized Lempert function *is defined by*

$$L_S^\Omega(z) := L_S(z)$$
$$:= \inf \left\{ \mathcal{S}(\varphi, (\alpha_j)_{1 \le j \le N}) : (\varphi, (\alpha_j)_{1 \le j \le N}) \text{ is admissible for } S, z \text{ in the old sense} \right\}.$$

Recall also that since the functional L did not enjoy monotonicity properties, another definition was given in [16].

Definition 7.2. *Let $S := \{(a_j, \Psi_j) : 1 \le j \le N\}$ and $S_1 := \{(a_j, \Psi_j^1) : 1 \le j \le N\}$ where $a_j \in \Omega$ and Ψ_j, Ψ_j^1 are local indicators. We define*

$$\tilde{L}_S(z) := \inf\{L_{S^1}(z) : \Psi_j^1 \ge \Psi_j + C_j, 1 \le j \le N\}.$$

Lemma 7.3. *If $S = \{(a_j, \Psi_j), 1 \le j \le N\}$, where the Ψ_j are elementary local indicators, then for any $z \in \Omega$, $\mathcal{L}_S(z) \le \tilde{L}_S(z)$.*

Proof. Since the functional $\mathcal{L}$ is monotonic by Theorem 3.2, it will be enough to show that $\mathcal{L}_S(z) \le L_S(z)$ for any system S. If we have a map φ which is admissible in the sense of Definition 7.1, we can take $A_j := \{\alpha_j\}$, and $\Psi_j(\varphi(\zeta) - a_j) \le \tau_j \log|\zeta - \alpha_j| + C_j$ implies that $m_{\varphi,a_j,\Psi_j}(\alpha_j) \ge \tau_j$, which by Definition 2.7 means that $m_{\varphi,a_j,\Psi_j}(\alpha_j) = \tau_j$. So that any such φ is admissible in the sense of Definition 2.8, and

$$\mathcal{S}(\varphi, (a_j)_{1\le j\le N}) = \mathcal{S}(\varphi, (A_j)_{1\le j\le N}),$$

and the desired inequality follows. □

We now return to the study of the example presented in [16]. Let us recall the notations. For $z \in \mathbb{D}^2$,

$$\Psi_0(z) := \max(\log|z_1|, \log|z_2|), \quad \Psi_V(z) := \max(\log|z_1|, 2\log|z_2|).$$

Here V stands for "vertical", for the obvious reasons : for $a \in \mathbb{D}^2$, $\Psi_j(\varphi(\zeta) - a) \le \tau_j \log|\zeta - \zeta_0| + C$ translates to ($\tau_0 = 1$, $\tau_V = 2$):

$$\begin{aligned} \varphi(\zeta_0) = a, &\quad \text{when} \quad j = 0, \\ \varphi(\zeta_0) = a, \varphi_1'(\zeta_0) = 0 &\quad \text{when} \quad j = V. \end{aligned}$$

For a, $b \in \mathbb{D}$ and $\varepsilon \in \mathbb{C}$, let

$$\begin{aligned} S_\varepsilon &:= \{((a,0),\Psi_0); ((b,0),\Psi_0); ((b,\varepsilon),\Psi_0); ((a,\varepsilon),\Psi_0)\} \\ S &:= \{((a,0),\Psi_V); ((b,0),\Psi_V)\}. \end{aligned}$$

Those are product set situations, and the Green functions are explicitly known. For $w \in \mathbb{D}$, denote by ϕ_w the unique involutive holomorphic automorphism of the disk which exchanges 0 and w:

$$\phi_w(\zeta) := \frac{w - \zeta}{1 - \zeta\bar{w}}.$$

Then

$$G_S(z_1, z_2) = \max\left(\log|\phi_a(z_1)\phi_b(z_2)|, 2\log|z_2|\right),$$
$$G_{S_\varepsilon}(z_1, z_2) = \max\left(\log|\phi_a(z_1)\phi_b(z_2)|, \log|z_2\phi_\varepsilon(z_2)|\right).$$

The following is proved in [16, p. 397].

Proposition 7.4. *If $b = -a$ and $|a|^2 < |\gamma| < |a|$, then $G_S(0,\gamma) < \tilde{L}_S(0,\gamma)$.*

It follows from our Theorem 3.3 that for any $z \in \mathbb{D}^2$, $\lim_{\varepsilon\to 0} L_{S_\varepsilon}(z) = \mathcal{L}_S(z)$, and in particular, using Lemma 7.3, we find again the result laboriously obtained in [15, Proposition 6.1]: $\limsup_{\varepsilon\to 0} L_{S_\varepsilon}(z) \le \tilde{L}_S(z)$. It is a consequence of [16, Theorem 5.1] (or equivalently [15, Theorem 6.2]) that for $b = -a$ and $|a|^{3/2} < |\gamma| < |a|$, then $\mathcal{L}_S(0,\gamma) > G_S(0,\gamma)$;

the motivation then was to obtain the counterexample $L_{S_\varepsilon}(0,\gamma) > G_{S_\varepsilon}(0,\gamma)$ for $|\varepsilon|$ small enough.

On the other hand, when $|\gamma| < |a|^{3/2}$, the old generalized Lempert function doesn't provide the correct limit of the single pole Lempert functions.

Proposition 7.5. *For $b = -a$ and $|a|^2 < |\gamma| < |a|^{3/2}$, $\mathcal{L}_S(0,\gamma) < \tilde{L}_S(0,\gamma)$.*

Proof. Since Proposition 7.4 implies that $\tilde{L}_S(0,\gamma) > G_S(0,\gamma) = 2\log|a|$, it will be enough to provide a mapping φ and sets A_1, A_2 admissible in the sense of Definition 2.8 such that $\mathcal{S}(\varphi; A_1, A_2) \le 2\log|a|$. We restrict ourselves to $a > 0$. We now choose $A_1 := \{\zeta_1, \zeta_4\}$, $A_2 := \{\zeta_2\}$, with

$$\zeta_2 := \sqrt{a}, \quad \zeta_1 := \phi_{\zeta_2}\left(\sqrt{\frac{2a}{1+a^2}}\right), \quad \zeta_4 := \phi_{\zeta_2}\left(-\sqrt{\frac{2a}{1+a^2}}\right),$$

and

$$\varphi_1(\zeta) := \phi_{-a}\left(-\phi_{\zeta_2}(\zeta)^2\right), \varphi_2(\zeta) := \frac{\gamma}{\zeta_1\zeta_2\zeta_4}\phi_{\zeta_1}(\zeta)\phi_{\zeta_2}(\zeta)\phi_{\zeta_4}(\zeta).$$

From those definitions it is clear that $\varphi_1(\mathbb{D}) \subset \mathbb{D}$ and that

$$\varphi_1(\zeta_2) = -a, \varphi_1'(\zeta_2) = 0; \quad \varphi_2(\zeta_j) = 0, \text{ for } j = 1,2,4.$$

Furthermore, using the involutivity of ϕ_{ζ_2},

$$\varphi_1(\zeta_1) = \varphi_1(\zeta_4) = \phi_{-a}\left(-\frac{2a}{1+a^2}\right) = \phi_{-a}\left(\phi_{-a}(a)\right) = a.$$

So the map φ hits the poles, and

$$m_{\varphi,(a,0),\Psi_V}(\zeta_1) \ge 1, \quad m_{\varphi,(a,0),\Psi_V}(\zeta_4) \ge 1, \quad m_{\varphi,(-a,0),\Psi_V}(\zeta_2) = 2.$$

To see that actually $m_{\varphi,(a,0),\Psi_V}(\zeta_j) = 1$, for $j = 1,4$, notice that, since φ_1 only admits one critical point, ζ_2, and since $\zeta_1 \ne \zeta_2$ and $\zeta_4 \ne \zeta_2$, we must have $\varphi_1'(\zeta_j) \ne 0$, $j = 1,4$.

Thus φ is admissible in the sense of Definition 2.8, and

$$\mathcal{S}(\varphi; A_1, A_2) = \log|\zeta_1| + 2\log|\zeta_2| + \log|\zeta_4| = \log|\zeta_1\zeta_4\zeta_2^2|.$$

We need to compute

$$\zeta_1\zeta_4 = \phi_{\sqrt{a}}\left(\sqrt{\frac{2a}{1+a^2}}\right)\cdot\phi_{\sqrt{a}}\left(-\sqrt{\frac{2a}{1+a^2}}\right)$$

$$= \frac{\sqrt{a}-\sqrt{\frac{2a}{1+a^2}}}{1-\sqrt{a}\sqrt{\frac{2a}{1+a^2}}}\cdot\frac{\sqrt{a}+\sqrt{\frac{2a}{1+a^2}}}{1+\sqrt{a}\sqrt{\frac{2a}{1+a^2}}} = \frac{a-\frac{2a}{1+a^2}}{1-a\frac{2a}{1+a^2}} = \phi_a\left(\phi_a(-a)\right) = -a.$$

From this we deduce $|\zeta_1\zeta_2\zeta_4| = a^{3/2} > |\gamma|$, and therefore $\varphi_2(\mathbb{D}) \subset \mathbb{D}$; and $|\zeta_1\zeta_4\zeta_2^2| = a^2$, therefore

$$\mathcal{S}(\varphi; A_1, A_2) \leq \log |\zeta_1\zeta_4\zeta_2^2| = 2\log |a|,$$

q.e.d. $\square$

References

1. E. Bedford, B. A. Taylor, *A new capacity for plurisubharmonic functions,* Acta Math. **149** (1982), 1–40.
2. M. Carlehed, J. Wiegerinck, *Le cône des fonctions plurisousharmoniques négatives et une conjecture de Coman,* Ann. Pol. Math. **80** (2003), 93–108.
3. D. Coman, *The pluricomplex Green function with two poles of the unit ball of* $\mathbb{C}^n$*,* Pacific J. Math. **194,** no 2, 257–283 (2000).
4. S. Dineen, *The Schwarz Lemma,* Clarendon Press, Oxford, 1989.
5. A. Edigarian, *Remarks on the pluricomplex Green function,* Univ. Iagel. Acta Math. **37** (1999), 159–164.
6. J. B. Garnett, *Bounded Analytic Functions,* Academic Press, Inc. New York-London, 1981.
7. M. Klimek, *Pluripotential Theory,* Oxford Science Publications, Oxford-New York-Tokyo, 1991.
8. P. Lelong and A. Rashkovskii, *Local indicators for plurisubharmonic functions,* J. Math. Pure Appl. **78**, 233–247 (1999).
9. L. Lempert, *La métrique de Kobayashi et la représentation des domains sur la boule,* Bull. Soc. Soc. Math. France **109**, 427–474 (1981).
10. N. Nikolov, P. Pflug, *The multipole Lempert function is monotone under inclusion of pole sets,* Mich. Math. J., **54**, no. 1,111–116 (2006).
11. N. Nikolov, W. Zwonek, *On the product property for the Lempert function,* Complex Var. Theory Appl. **50**, no. 12, 939–952 (2005).
12. A. Rashkovskii, *Newton numbers and residual measures of plurisubharmonic functions,* Ann. Polon. Math. **75** (2000), 213–231.
13. A. Rashkovskii, R. Sigurdsson, *Green functions with singularities along complex spaces,* Internat. J. Math. **16** (2005), no. 4, 333–355.
14. P. J. Thomas, *An example of limit of Lempert Functions,* Viet Nam J. Math. 35, no. 3, 1–14 (2007).
15. P. J. Thomas, N. V. Trao, *Pluricomplex Green and Lempert functions for equally weighted poles.* Prépublication no. 240 du Laboratoire de Mathématiques Emile Picard, Université Paul Sabatier, Toulouse, France, march 2002 (25 pp.). Available as an e-print on math arXiv: http://xxx.lanl.gov/abs/math.CV/0206214.

16. P. J. Thomas, N. V. Trao, *Pluricomplex Green and Lempert functions for equally weighted poles.* Ark. Mat. 41, no. 2, 381–400 (2003).

17. F. Wikström, *Jensen Measures, Duality and Plurisubharmonic Green Functions*, Doctoral thesis No 18, Umeå University, 1999.

18. F. Wikström, *Non-linearity of the pluricomplex Green function,* Proc. Amer. Math. Soc. 129, no. 4, 1051–1056 (2001).

Pascal J. Thomas
Institut de Mathématiques de Toulouse
CNRS UMR 5219
UFR MIG
Université Paul Sabatier
F-31062 TOULOUSE CEDEX 9
France
pthomas@math.univ-toulouse.fr

Nguyen Van Trao
Department of Mathematics
Dai Hoc Su Pham 1 (Pedagogical Institute of Hanoi)
Cau Giay, Tu Liem
Ha Noi
Viet Nam
ngvtrao@yahoo.com